SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3492 ★ . [2009 : 515, 518; 2010 : 558] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let P be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB, \angle PBC, \angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that P is its centroid.

Solution by Tomasz Cieśla, student, University of Warsaw, Poland.

The claim is not true; we shall see that there are infinitely many counterexamples. Consider points A', B, C on a circle with centre P such that $\angle PA'B = \angle PBA' = \angle PBC = \angle PCB = \arccos\sqrt{\frac{2}{3}}$. For $\alpha \in (0, \pi)$ the rotation about line PB through the angle α maps point A' into a point A (preserving angles $PA'B$ and PBA'). Let D be the reflection of point B in the plane PCA (so that $\angle PCD = \angle PCB$ and $\angle PDA = \angle PBA = \angle PBA'$.) Then the tetrahedron $ABCD$ satisfies

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \arccos\sqrt{\frac{2}{3}},$$

as required. Note that we can choose α so that $ABCD$ is not regular; in fact, there is only one value of α which produces a regular tetrahedron (occurring when $\angle PAC = \arccos\sqrt{\frac{2}{3}}$). Values of α close to that ensure that P lies in the interior of $ABCD$ and, therefore, provide counterexamples to the problem as it was stated.

No other correspondence about this problem has been received.

3631. [2011: 171, 173] *Proposed by Michel Bataille, Rouen, France.*

Let $\{x_n\}$ be the sequence satisfying $x_0 = 1$, $x_1 = 2011$, and $x_{n+2} = 2012x_{n+1} - x_n$ for all nonnegative integers n. Prove that

$$\frac{(2010 + x_n^2 + x_{n+1}^2)(2010 + x_{n+2}^2 + x_{n+3}^2)}{(2010 + x_{n+1}^2)(2010 + x_{n+2}^2)}$$

is independent of n.

Solution by Arkady Alt, San Jose, CA, USA.

More generally, let a be an integer and let $\{x_n\}$ be determined by $x_{-1} = x_0 = 1$, $x_1 = a + 1$ and $x_{n+2} = (a + 2)x_{n+1} - x_n$ for $n \geq 0$. Since

$$\begin{aligned} x_n x_{n+2} - x_{n+1}^2 &= x_n[(a+2)x_{n+1} - x_n] - x_{n+1}^2 = x_{n+1}[(a+2)x_n - x_{n+1}] - x_n^2 \\ &= x_{n-1}x_{n+1} - x_n^2, \end{aligned}$$

it follows that

$$x_n x_{n+2} - x_{n+1}^2 = x_0 x_2 - x_1^2 = (a^2 + 3a + 1) - (a+1)^2 = a$$

for $n \geq 0$. Therefore, for $n \geq 0$,

$$\begin{aligned}\frac{(a + x_n^2 + x_{n+1}^2)(a + x_{n+2}^2 + x_{n+3}^2)}{(a + x_{n+1}^2)(a + x_{n+2}^2)} &= \frac{(x_{n-1}x_{n+1} + x_{n+1}^2)(x_{n+1}x_{n+3} + x_{n+3}^2)}{(x_n x_{n+2})(x_{n+1}x_{n+3})} \\ &= \frac{x_{n+1}(x_{n-1} + x_{n+1})x_{n+3}(x_{n+1} + x_{n+3})}{(x_{n+1}x_{n+3})(x_n x_{n+2})} \\ &= \frac{[(a+2)x_n][(a+2)x_{n+2}]}{x_n x_{n+2}} = (a+2)^2.\end{aligned}$$

Taking $a = 2010$ yields the value 2012^2 for the expression in the problem.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comăneşti, Romania; and the proposer. Apostolopoulos and the proposer had solutions similar to the one given, while the remaining solvers solved the recursion and used the formula for the general term. One additional person simply gave the answer with no justification.

3632. [2011: 171, 173] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let k be a real number such that $0 \leq k \leq 56$. Prove that the equation below has exactly two real solutions:

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; k(x^2 - 7x) + 720\,.$$

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

We prove that the result holds as long as $k < 945/16 = 59.0625$. The difference $P(x)$ betweeen the two sides of the equation is given by

$$\begin{aligned}P(x) &= [(x-1)(x-6)][(x-2)(x-5)][(x-3)(x-4)] - k(x^2 - 7x) - 720 \\ &= [(x^2 - 7x) + 6][(x^2 - 7x) + 10][(x^2 - 7x) + 12] - k(x^2 - 7x) - 720 \\ &= (x^2 - 7x)[(x^2 - 7x)^2 + 28(x^2 - 7x) + (252 - k)] \\ &= \frac{1}{16}(x^2 - 7x)\{[(2x-7)^2 - 49]^2 + 112[(2x-7)^2 - 49] + (4032 - 16k)\} \\ &= \frac{1}{16}x(x-7)[(2x-7)^4 + 14(2x-7)^2 + (945 - 16k)].\end{aligned}$$

When $k < 945/16$, the final factor is positive for all real x, so that the only real roots of $P(x)$ are 0 and 7.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX,